

A GENERAL CONVERGENCE RESULT FOR THE RICCI FLOW IN HIGHER DIMENSIONS

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1. INTRODUCTION

In this paper, we study the longterm behavior of the Ricci flow in higher dimensions. A one-parameter family of metrics $g(t)$ is a solution to the Ricci flow if

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)},$$

where $\operatorname{Ric}_{g(t)}$ denotes the Ricci tensor of $g(t)$ (cf. [3]). Moreover, $g(t)$ is a solution to the normalized Ricci flow if

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)} + \frac{2}{n} r_{g(t)} g(t),$$

where $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$. In a joint work with R. Schoen, we proved the following theorem:

Theorem 1 ([2], Theorem 3). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$. Assume that*

$$(1) \quad R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} > 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0 exists for all time and converges to a constant curvature metric as $t \rightarrow \infty$.

An immediate consequence of Theorem 1 is the Differentiable Sphere Theorem: if (M, g_0) has strictly $1/4$ -pinched sectional curvatures, then M is diffeomorphic to a spherical space form. We refer to [2] for a discussion of the history of this problem.

In this paper, we weaken the curvature assumption in Theorem 1. Our main result is the following:

Theorem 2. *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$. Assume that*

$$(2) \quad R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0 exists for all time and converges to a constant curvature metric as $t \rightarrow \infty$.

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C. Böhm and B. Wilking [1] have shown that the normalized Ricci flow deforms metrics with 2-positive curvature operator to constant curvature metrics. It is easy to see that every manifold with 2-positive curvature operator satisfies condition (2). Hence, the main theorem in [1] is a subcase of Theorem 2.

The conditions (1) and (2) are closely related to the notion of positive isotropic curvature. To explain this, suppose that M is a Riemannian manifold of dimension $n \geq 4$. We say that M has nonnegative isotropic curvature if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ (cf. [6], [7]). The product $M \times \mathbb{R}$ has nonnegative isotropic curvature if and only if

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$ (see Proposition 4 below). Similarly, the product $M \times \mathbb{R}^2$ has nonnegative isotropic curvature if and only if

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [-1, 1]$ (see [2], Proposition 21).

The curvature conditions (1) and (2) are void in dimension less than 4. However, the condition that $M \times \mathbb{R}$ has nonnegative isotropic curvature makes sense for all $n \geq 3$, and the condition that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature makes sense for all $n \geq 2$. A three-manifold M has nonnegative Ricci curvature if and only if $M \times \mathbb{R}$ has nonnegative isotropic curvature. Moreover, a three-manifold M has nonnegative sectional curvature if and only if $M \times \mathbb{R}^2$ has nonnegative isotropic curvature. Thus, Theorem 2 can be viewed as a generalization of a theorem of R. Hamilton on three-manifolds with positive Ricci curvature (see [3]). Combining the two results, we obtain:

Theorem 3. *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 3$. If $(M, g_0) \times \mathbb{R}$ has positive isotropic curvature, then the normalized Ricci flow with initial metric g_0 exists for all time and converges to a constant curvature metric as $t \rightarrow \infty$.*

R. Hamilton [5] has shown that the Ricci flow preserves positive isotropic curvature in dimension 4. Moreover, Hamilton proved that, in dimension 4, a solution to the Ricci flow with positive isotropic curvature develops only "neck-like" singularities. More recently, it was shown that positive isotropic curvature is preserved by the Ricci flow in all dimensions. This result was proved independently in [2] and [8]. It is an open question whether the analysis of singularities in [5] carries over to higher dimensions. We hope that Theorem 3 will shed light on this question.

In Section 2, we consider the condition that $M \times \mathbb{R}$ has nonnegative isotropic curvature. This condition defines a convex cone \tilde{C} in the space of algebraic curvature operators, which is preserved by the Hamilton ODE.

In Section 3, we consider the condition that $M \times S^2(1)$ has nonnegative isotropic curvature. This defines a convex set E in the space of algebraic curvature operators. It is easy to see that $\hat{C} \subset E \subset \tilde{C}$, where \hat{C} denotes the cone introduced in [2]. Using results from [2], we show that the set E is invariant under the Hamilton ODE. This fact is the main ingredient in the proof of Theorem 2.

In Section 4, we complete the proof of Theorem 2 by constructing a suitable pinching set for the Hamilton ODE.

2. THE CONE \tilde{C}

Let R be an algebraic curvature operator on \mathbb{R}^n . We define an algebraic curvature operator \tilde{R} on $\mathbb{R}^n \times \mathbb{R}$ by

$$\tilde{R}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) = R(v_1, v_2, v_3, v_4)$$

for all vectors $\tilde{v}_j = (v_j, x_j) \in \mathbb{R}^n \times \mathbb{R}$. We denote by \tilde{C} the set of all algebraic curvature operators on \mathbb{R}^n with the property that \tilde{R} has nonnegative isotropic curvature:

$$\tilde{C} = \{R \in S_B^2(\mathfrak{so}(n)) : \tilde{R} \text{ has nonnegative isotropic curvature}\}.$$

Clearly, \tilde{C} is closed, convex, and $O(n)$ -invariant. Moreover, it follows from the results in [2] that \tilde{C} is invariant under the Hamilton ODE $\frac{d}{dt}R = Q(R)$. The cone \tilde{C} can be characterized as follows:

Proposition 4. *Let R be an algebraic curvature operator on \mathbb{R}^n , and let \tilde{R} be the induced curvature operator on $\mathbb{R}^n \times \mathbb{R}$. The following statements are equivalent:*

- (i) \tilde{R} has nonnegative isotropic curvature.
- (ii) For all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$, we have

$$\begin{aligned} & R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ & + R(e_2, e_3, e_2, e_3) + \lambda^2 R(e_2, e_4, e_2, e_4) - 2\lambda R(e_1, e_2, e_3, e_4) \geq 0. \end{aligned}$$

Proof. Assume first that \tilde{R} has nonnegative isotropic curvature. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal four-frame in \mathbb{R}^n , and let $\lambda \in [-1, 1]$. We define

$$\begin{aligned} \tilde{e}_1 &= (e_1, 0) & \tilde{e}_2 &= (e_2, 0) \\ \tilde{e}_3 &= (e_3, 0) & \tilde{e}_4 &= (\lambda e_4, \sqrt{1 - \lambda^2}). \end{aligned}$$

Since \tilde{R} has nonnegative isotropic curvature, we have

$$\begin{aligned}
0 &\leq \tilde{R}(\tilde{e}_1, \tilde{e}_3, \tilde{e}_1, \tilde{e}_3) + \tilde{R}(\tilde{e}_1, \tilde{e}_4, \tilde{e}_1, \tilde{e}_4) \\
&\quad + \tilde{R}(\tilde{e}_2, \tilde{e}_3, \tilde{e}_2, \tilde{e}_3) + \tilde{R}(\tilde{e}_2, \tilde{e}_4, \tilde{e}_2, \tilde{e}_4) - 2\tilde{R}(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4) \\
&= R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\
&\quad + R(e_2, e_3, e_2, e_3) + \lambda^2 R(e_2, e_4, e_2, e_4) - 2\lambda R(e_1, e_2, e_3, e_4),
\end{aligned}$$

as claimed.

Conversely, assume that (ii) holds. We claim that \tilde{R} has nonnegative isotropic curvature. Let $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ be an orthonormal four-frame in $\mathbb{R}^n \times \mathbb{R}$. We write $\tilde{e}_j = (v_j, x_j)$, where $v_j \in \mathbb{R}^n$ and $x_j \in \mathbb{R}$. Moreover, we define

$$\begin{aligned}
\varphi &= v_1 \wedge v_3 + v_4 \wedge v_2 \\
\psi &= v_1 \wedge v_4 + v_2 \wedge v_3.
\end{aligned}$$

Clearly, $\varphi \wedge \varphi = \psi \wedge \psi$ and $\varphi \wedge \psi = 0$. Using the relation $\langle v_i, v_j \rangle + x_i x_j = \delta_{ij}$, we obtain

$$\begin{aligned}
|\varphi|^2 - |\psi|^2 &= |v_1 \wedge v_3|^2 + |v_4 \wedge v_2|^2 - |v_1 \wedge v_4|^2 - |v_2 \wedge v_3|^2 \\
&\quad + 2\langle v_1 \wedge v_3, v_4 \wedge v_2 \rangle - 2\langle v_1 \wedge v_4, v_2 \wedge v_3 \rangle \\
&= (|v_1|^2 - |v_2|^2)(|v_3|^2 - |v_4|^2) - 4\langle v_1, v_2 \rangle \langle v_3, v_4 \rangle \\
&\quad - (\langle v_1, v_3 \rangle - \langle v_2, v_4 \rangle)^2 + (\langle v_1, v_4 \rangle^2 + \langle v_2, v_3 \rangle)^2 \\
&= (x_1^2 - x_2^2)(x_3^2 - x_4^2) - 4x_1 x_2 x_3 x_4 \\
&\quad - (x_1 x_3 - x_2 x_4)^2 + (x_1 x_4 + x_2 x_3)^2 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\langle \varphi, \psi \rangle &= \langle v_1 \wedge v_3, v_1 \wedge v_4 \rangle + \langle v_1 \wedge v_3, v_2 \wedge v_3 \rangle \\
&\quad + \langle v_4 \wedge v_2, v_1 \wedge v_4 \rangle + \langle v_4 \wedge v_2, v_2 \wedge v_3 \rangle \\
&= (|v_1|^2 - |v_2|^2) \langle v_3, v_4 \rangle + (|v_3|^2 - |v_4|^2) \langle v_1, v_2 \rangle \\
&\quad - (\langle v_1, v_3 \rangle - \langle v_2, v_4 \rangle)(\langle v_1, v_4 \rangle + \langle v_2, v_3 \rangle) \\
&= (x_1^2 - x_2^2) x_3 x_4 + (x_3^2 - x_4^2) x_1 x_2 \\
&\quad - (x_1 x_3 - x_2 x_4)(x_1 x_4 + x_2 x_3) \\
&= 0.
\end{aligned}$$

By Lemma 19 in [2], we can find an orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ in \mathbb{R}^n and real numbers a_1, a_2, b_1, b_2 such that $a_1^2 + a_2^2 = b_1^2 + b_2^2$, $a_1 a_2 = b_1 b_2$, and

$$\begin{aligned}
\varphi &= a_1 e_1 \wedge e_3 + a_2 e_4 \wedge e_2 \\
\psi &= b_1 e_1 \wedge e_4 + b_2 e_2 \wedge e_3.
\end{aligned}$$

Clearly, $(a_1^2 - b_1^2)(a_2^2 - b_2^2) = 0$. Without loss of generality, we may assume that $a_1^2 = b_1^2$. (Otherwise, we replace $\{e_1, e_2, e_3, e_4\}$ by $\{e_3, e_4, e_1, e_2\}$.) This

implies $a_2^2 = b_1^2$. Using the first Bianchi identity, we obtain

$$\begin{aligned} R(\varphi, \varphi) + R(\psi, \psi) &= a_1^2 R(e_1, e_3, e_1, e_3) + a_2^2 R(e_1, e_4, e_1, e_4) \\ &\quad + a_1^2 R(e_2, e_3, e_2, e_3) + a_2^2 R(e_2, e_4, e_2, e_4) \\ &\quad - 2a_1 a_2 R(e_1, e_2, e_3, e_4). \end{aligned}$$

The condition (ii) implies that the right hand side is nonnegative. Thus, we conclude that

$$\begin{aligned} &\tilde{R}(\tilde{e}_1, \tilde{e}_3, \tilde{e}_1, \tilde{e}_3) + \tilde{R}(\tilde{e}_1, \tilde{e}_4, \tilde{e}_1, \tilde{e}_4) \\ &\quad + \tilde{R}(\tilde{e}_2, \tilde{e}_3, \tilde{e}_2, \tilde{e}_3) + \tilde{R}(\tilde{e}_2, \tilde{e}_4, \tilde{e}_2, \tilde{e}_4) - 2\tilde{R}(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4) \\ &= R(\varphi, \varphi) + R(\psi, \psi) \geq 0. \end{aligned}$$

Hence, \tilde{R} has nonnegative isotropic curvature.

3. A NEW INVARIANT CURVATURE CONDITION

Let R be an algebraic curvature operator on \mathbb{R}^n . Following Hamilton [4], we define

$$Q(R)_{ijkl} = R_{ijpq} R_{klpq} + 2R_{ipkq} R_{jplq} - 2R_{iplq} R_{jpkq}.$$

It is straightforward to verify that $Q(R)$ is an algebraic curvature tensor. As in [4], we write $Q(R) = R^2 + R^\#$, where R^2 and $R^\#$ are defined by

$$\begin{aligned} (R^2)_{ijkl} &= R_{ijpq} R_{klpq} \\ (R^\#)_{ijkl} &= 2R_{ipkq} R_{jplq} - 2R_{iplq} R_{jpkq}. \end{aligned}$$

Note that R^2 and $R^\#$ do not satisfy the first Bianchi identity, but $R^2 + R^\#$ does. The following lemma is a consequence of Corollary 10 in [2], and plays a key role in our analysis:

Lemma 5. *Let R be an algebraic curvature operator on \mathbb{R}^n with nonnegative isotropic curvature. Moreover, suppose that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame in \mathbb{R}^n satisfying*

$$\begin{aligned} &R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4) \\ (3) \quad &+ R(e_2, e_3, e_2, e_3) + R(e_2, e_4, e_2, e_4) - 2R(e_1, e_2, e_3, e_4) = 0. \end{aligned}$$

Then

$$\begin{aligned} &R^\#(e_1, e_3, e_1, e_3) + R^\#(e_1, e_4, e_1, e_4) \\ (4) \quad &+ R^\#(e_2, e_3, e_2, e_3) + R^\#(e_2, e_4, e_2, e_4) \\ &+ 2R^\#(e_1, e_3, e_4, e_2) + 2R^\#(e_1, e_4, e_2, e_3) \geq 0. \end{aligned}$$

Proof. We extend $\{e_1, e_2, e_3, e_4\}$ to an orthonormal basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n . Using the first Bianchi identity, we obtain

$$\begin{aligned} & R^\#(e_1, e_3, e_4, e_2) + R^\#(e_1, e_4, e_2, e_3) \\ &= 2R(e_1, e_p, e_4, e_q)R(e_3, e_p, e_2, e_q) - 2R(e_1, e_p, e_3, e_q)R(e_4, e_p, e_2, e_q) \\ &+ 2R(e_1, e_p, e_2, e_q)R(e_4, e_p, e_3, e_q) - 2R(e_1, e_p, e_2, e_q)R(e_3, e_p, e_4, e_q) \\ &= 2R(e_1, e_p, e_4, e_q)R(e_3, e_p, e_2, e_q) - 2R(e_1, e_p, e_3, e_q)R(e_4, e_p, e_2, e_q) \\ &- R(e_1, e_2, e_p, e_q)R(e_3, e_4, e_p, e_q). \end{aligned}$$

This implies

$$\begin{aligned} & R^\#(e_1, e_3, e_1, e_3) + R^\#(e_1, e_4, e_1, e_4) \\ &+ R^\#(e_2, e_3, e_2, e_3) + R^\#(e_2, e_4, e_2, e_4) \\ &+ 2R^\#(e_1, e_3, e_4, e_2) + 2R^\#(e_1, e_4, e_2, e_3) \\ &= 2R(e_1, e_p, e_1, e_q)R(e_3, e_p, e_3, e_q) - 2R(e_1, e_p, e_3, e_q)R(e_3, e_p, e_1, e_q) \\ &+ 2R(e_1, e_p, e_1, e_q)R(e_4, e_p, e_4, e_q) - 2R(e_1, e_p, e_4, e_q)R(e_4, e_p, e_1, e_q) \\ &+ 2R(e_2, e_p, e_2, e_q)R(e_3, e_p, e_3, e_q) - 2R(e_2, e_p, e_3, e_q)R(e_3, e_p, e_2, e_q) \\ &+ 2R(e_2, e_p, e_2, e_q)R(e_4, e_p, e_4, e_q) - 2R(e_2, e_p, e_4, e_q)R(e_4, e_p, e_2, e_q) \\ &+ 4R(e_1, e_p, e_4, e_q)R(e_3, e_p, e_2, e_q) - 4R(e_1, e_p, e_3, e_q)R(e_4, e_p, e_2, e_q) \\ &- 2R(e_1, e_2, e_p, e_q)R(e_3, e_4, e_p, e_q). \end{aligned}$$

Rearranging terms yields

$$\begin{aligned} & R^\#(e_1, e_3, e_1, e_3) + R^\#(e_1, e_4, e_1, e_4) \\ &+ R^\#(e_2, e_3, e_2, e_3) + R^\#(e_2, e_4, e_2, e_4) \\ &+ 2R^\#(e_1, e_3, e_4, e_2) + 2R^\#(e_1, e_4, e_2, e_3) \\ &= 2(R(e_1, e_p, e_1, e_q) + R(e_2, e_p, e_2, e_q))(R(e_3, e_p, e_3, e_q) + R(e_4, e_p, e_4, e_q)) \\ &- 2R(e_1, e_2, e_p, e_q)R(e_3, e_4, e_p, e_q) \\ &- 2(R(e_1, e_p, e_3, e_q) + R(e_2, e_p, e_4, e_q))(R(e_3, e_p, e_1, e_q) + R(e_4, e_p, e_2, e_q)) \\ &+ 2(R(e_1, e_p, e_4, e_q) - R(e_2, e_p, e_3, e_q))(R(e_4, e_p, e_1, e_q) - R(e_3, e_p, e_2, e_q)), \end{aligned}$$

and the right hand side is nonnegative by Corollary 10 in [2].

Given any algebraic curvature operator R on \mathbb{R}^n , we define an algebraic curvature operator S on $\mathbb{R}^n \times \mathbb{R}^2$ by

$$S(\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4) = R(v_1, v_2, v_3, v_4) + \langle x_1, x_3 \rangle \langle x_2, x_4 \rangle - \langle x_1, x_4 \rangle \langle x_2, x_3 \rangle$$

for all vectors $\hat{v}_j = (v_j, x_j) \in \mathbb{R}^n \times \mathbb{R}^2$. A straightforward calculation yields:

Lemma 6. *Let R be an algebraic curvature operator on \mathbb{R}^n , and let S be the induced curvature operator on $\mathbb{R}^n \times \mathbb{R}^2$. Then*

$$S^\#(\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4) = R^\#(v_1, v_2, v_3, v_4)$$

for all vectors $\hat{v}_j = (v_j, x_j) \in \mathbb{R}^n \times \mathbb{R}^2$.

Let E be the set of all algebraic curvature operators on \mathbb{R}^n with the property that the induced curvature operator S on $\mathbb{R}^n \times \mathbb{R}^2$ has nonnegative isotropic curvature:

$$E = \{R \in S_B^2(\mathfrak{so}(n)) : S \text{ has nonnegative isotropic curvature}\}$$

It is easy to see that E is closed, convex, and $O(n)$ -invariant.

Proposition 7. *Let R be an algebraic curvature operator on \mathbb{R}^n , and let S be the induced curvature operator on $\mathbb{R}^n \times \mathbb{R}^2$. The following statements are equivalent:*

- (i) *S has nonnegative isotropic curvature.*
- (ii) *For all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [-1, 1]$, we have*

$$\begin{aligned} & R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ & + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) \\ & - 2\lambda\mu R(e_1, e_2, e_3, e_4) + (1 - \lambda^2)(1 - \mu^2) \geq 0. \end{aligned}$$

Proof. Assume first that S has nonnegative isotropic curvature. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal four-frame in \mathbb{R}^n , and let $\lambda, \mu \in [-1, 1]$. We define

$$\begin{aligned} \hat{e}_1 &= (e_1, 0, 0) & \hat{e}_2 &= (\mu e_2, 0, \sqrt{1 - \mu^2}) \\ \hat{e}_3 &= (e_3, 0, 0) & \hat{e}_4 &= (\lambda e_4, \sqrt{1 - \lambda^2}, 0). \end{aligned}$$

Clearly, the vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ form an orthonormal four-frame in $\mathbb{R}^n \times \mathbb{R}^2$. Since S has nonnegative isotropic curvature, we have

$$\begin{aligned} 0 &\leq S(\hat{e}_1, \hat{e}_3, \hat{e}_1, \hat{e}_3) + S(\hat{e}_1, \hat{e}_4, \hat{e}_1, \hat{e}_4) \\ &+ S(\hat{e}_2, \hat{e}_3, \hat{e}_2, \hat{e}_3) + S(\hat{e}_2, \hat{e}_4, \hat{e}_2, \hat{e}_4) - 2S(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4) \\ &= R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ &+ \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) \\ &- 2\lambda\mu R(e_1, e_2, e_3, e_4) + (1 - \lambda^2)(1 - \mu^2), \end{aligned}$$

as claimed.

Conversely, assume that (ii) holds. We claim that S has nonnegative isotropic curvature. Let $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ be an orthonormal four-frame in $\mathbb{R}^n \times \mathbb{R}^2$. We write $\hat{e}_j = (v_j, x_j)$, where $v_j \in \mathbb{R}^n$ and $x_j \in \mathbb{R}^2$. Let V be a four-dimensional subspace of \mathbb{R}^n containing $\{v_1, v_2, v_3, v_4\}$. We define

$$\begin{aligned} \varphi &= v_1 \wedge v_3 + v_4 \wedge v_2 \in \wedge^2 V \\ \psi &= v_1 \wedge v_4 + v_2 \wedge v_3 \in \wedge^2 V. \end{aligned}$$

Clearly, $\varphi \wedge \varphi = \psi \wedge \psi$ and $\varphi \wedge \psi = 0$. By Lemma 20 in [2], there exist an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of V and real numbers $a_1, a_2, b_1, b_2, \theta$ such

that $a_1 a_2 = b_1 b_2$ and

$$\begin{aligned}\tilde{\varphi} &:= \cos \theta \varphi + \sin \theta \psi = a_1 e_1 \wedge e_3 + a_2 e_4 \wedge e_2 \\ \tilde{\psi} &:= -\sin \theta \varphi + \cos \theta \psi = b_1 e_1 \wedge e_4 + b_2 e_2 \wedge e_3.\end{aligned}$$

This implies

$$\begin{aligned}R(\varphi, \varphi) + R(\psi, \psi) &= R(\tilde{\varphi}, \tilde{\varphi}) + R(\tilde{\psi}, \tilde{\psi}) \\ &= a_1^2 R(e_1, e_3, e_1, e_3) + b_1^2 R(e_1, e_4, e_1, e_4) \\ &\quad + b_2^2 R(e_2, e_3, e_2, e_3) + a_2^2 R(e_2, e_4, e_2, e_4) \\ &\quad - 2a_1 a_2 R(e_1, e_2, e_3, e_4).\end{aligned}$$

Using the identity $\langle v_i, v_j \rangle + \langle x_i, x_j \rangle = \delta_{ij}$, we obtain

$$\begin{aligned}|\varphi|^2 - |\psi|^2 &= (|v_1|^2 - |v_2|^2)(|v_3|^2 - |v_4|^2) - 4 \langle v_1, v_2 \rangle \langle v_3, v_4 \rangle \\ &\quad - (\langle v_1, v_3 \rangle - \langle v_2, v_4 \rangle)^2 + (\langle v_1, v_4 \rangle + \langle v_2, v_3 \rangle)^2 \\ &= (|x_1|^2 - |x_2|^2)(|x_3|^2 - |x_4|^2) - 4 \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \\ &\quad - (\langle x_1, x_3 \rangle - \langle x_2, x_4 \rangle)^2 + (\langle x_1, x_4 \rangle + \langle x_2, x_3 \rangle)^2 \\ &= |x_1 \wedge x_3 + x_4 \wedge x_2|^2 - |x_1 \wedge x_4 + x_2 \wedge x_3|^2\end{aligned}$$

and

$$\begin{aligned}\langle \varphi, \psi \rangle &= (|v_1|^2 - |v_2|^2) \langle v_3, v_4 \rangle + (|v_3|^2 - |v_4|^2) \langle v_1, v_2 \rangle \\ &\quad - (\langle v_1, v_3 \rangle - \langle v_2, v_4 \rangle) (\langle v_1, v_4 \rangle + \langle v_2, v_3 \rangle) \\ &= (|x_1|^2 - |x_2|^2) \langle x_3, x_4 \rangle + (|x_3|^2 - |x_4|^2) \langle x_1, x_2 \rangle \\ &\quad - (\langle x_1, x_3 \rangle - \langle x_2, x_4 \rangle) (\langle x_1, x_4 \rangle + \langle x_2, x_3 \rangle) \\ &= \langle x_1 \wedge x_3 + x_4 \wedge x_2, x_1 \wedge x_4 + x_2 \wedge x_3 \rangle.\end{aligned}$$

From this we deduce that

$$\begin{aligned}&(|x_1 \wedge x_3 + x_4 \wedge x_2|^2 + |x_1 \wedge x_4 + x_2 \wedge x_3|^2)^2 \\ &= (|x_1 \wedge x_3 + x_4 \wedge x_2|^2 - |x_1 \wedge x_4 + x_2 \wedge x_3|^2)^2 \\ &\quad + 4|x_1 \wedge x_3 + x_4 \wedge x_2|^2 |x_1 \wedge x_4 + x_2 \wedge x_3|^2 \\ &\geq (|x_1 \wedge x_3 + x_4 \wedge x_2|^2 - |x_1 \wedge x_4 + x_2 \wedge x_3|^2)^2 \\ &\quad + 4 \langle x_1 \wedge x_3 + x_4 \wedge x_2, x_1 \wedge x_4 + x_2 \wedge x_3 \rangle^2 \\ &= (|\varphi|^2 - |\psi|^2)^2 + 4 \langle \varphi, \psi \rangle^2 \\ &= (|\tilde{\varphi}|^2 - |\tilde{\psi}|^2)^2 + 4 \langle \tilde{\varphi}, \tilde{\psi} \rangle^2 \\ &= (a_1^2 + a_2^2 - b_1^2 - b_2^2)^2.\end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned}
& R(\varphi, \varphi) + R(\psi, \psi) + |x_1 \wedge x_3 + x_4 \wedge x_2|^2 + |x_1 \wedge x_4 + x_2 \wedge x_3|^2 \\
& \geq a_1^2 R(e_1, e_3, e_1, e_3) + b_1^2 R(e_1, e_4, e_1, e_4) \\
& + b_2^2 R(e_2, e_3, e_2, e_3) + a_2^2 R(e_2, e_4, e_2, e_4) \\
& - 2a_1a_2 R(e_1, e_2, e_3, e_4) + |a_1^2 + a_2^2 - b_1^2 - b_2^2|.
\end{aligned}$$

The condition (ii) implies that the right hand side is nonnegative. Thus, we conclude that

$$\begin{aligned}
& S(\hat{e}_1, \hat{e}_3, \hat{e}_1, \hat{e}_3) + S(\hat{e}_1, \hat{e}_4, \hat{e}_1, \hat{e}_4) \\
& + S(\hat{e}_2, \hat{e}_3, \hat{e}_2, \hat{e}_3) + S(\hat{e}_2, \hat{e}_4, \hat{e}_2, \hat{e}_4) - 2S(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4) \\
& = R(\varphi, \varphi) + R(\psi, \psi) + |x_1 \wedge x_3 + x_4 \wedge x_2|^2 + |x_1 \wedge x_4 + x_2 \wedge x_3|^2 \geq 0,
\end{aligned}$$

as claimed.

We next consider the cone \hat{C} introduced in [2]. Moreover, we denote by I the curvature operator of the standard sphere, i.e. $I_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. Using Proposition 21 in [2], we obtain:

Corollary 8. *If $R \in E$, then $R \in \tilde{C}$ and $R + I \in \hat{C}$. Moreover, we have $E + \hat{C} = E$.*

We claim that the set E is invariant under the ODE $\frac{d}{dt}R = Q(R)$. This is a consequence of the following algebraic fact:

Proposition 9. *Let $R \in E$ be an algebraic curvature operator on \mathbb{R}^n . Moreover, let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal four-frame in \mathbb{R}^n , and let $\lambda, \mu \in [-1, 1]$. If*

$$\begin{aligned}
& R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\
(5) \quad & + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) \\
& - 2\lambda\mu R(e_1, e_2, e_3, e_4) + (1 - \lambda^2)(1 - \mu^2) = 0,
\end{aligned}$$

then we have

$$\begin{aligned}
& Q(R)(e_1, e_3, e_1, e_3) + \lambda^2 Q(R)(e_1, e_4, e_1, e_4) \\
(6) \quad & + \mu^2 Q(R)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 Q(R)(e_2, e_4, e_2, e_4) \\
& - 2\lambda\mu Q(R)(e_1, e_2, e_3, e_4) \geq 0.
\end{aligned}$$

Proof. Let S be the curvature operator on $\mathbb{R}^n \times \mathbb{R}^2$ associated with R . We define an orthonormal four-frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ in $\mathbb{R}^n \times \mathbb{R}^2$ by

$$\begin{aligned}
\hat{e}_1 &= (e_1, 0, 0) & \hat{e}_2 &= (\mu e_2, 0, \sqrt{1 - \mu^2}) \\
\hat{e}_3 &= (e_3, 0, 0) & \hat{e}_4 &= (\lambda e_4, \sqrt{1 - \lambda^2}, 0).
\end{aligned}$$

By assumption, S has nonnegative isotropic curvature. Moreover, it follows from (5) that

$$\begin{aligned} & S(\hat{e}_1, \hat{e}_3, \hat{e}_1, \hat{e}_3) + S(\hat{e}_1, \hat{e}_4, \hat{e}_1, \hat{e}_4) \\ & + S(\hat{e}_2, \hat{e}_3, \hat{e}_2, \hat{e}_3) + S(\hat{e}_2, \hat{e}_4, \hat{e}_2, \hat{e}_4) - 2S(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4) = 0. \end{aligned}$$

Hence, Lemma 5 implies that

$$\begin{aligned} & S^\#(\hat{e}_1, \hat{e}_3, \hat{e}_1, e_3) + S^\#(\hat{e}_1, \hat{e}_4, \hat{e}_1, e_4) \\ & + S^\#(\hat{e}_2, \hat{e}_3, \hat{e}_2, e_3) + S^\#(\hat{e}_2, \hat{e}_4, \hat{e}_2, e_4) \\ & + 2S^\#(\hat{e}_1, \hat{e}_3, \hat{e}_4, e_2) + 2S^\#(\hat{e}_1, \hat{e}_4, \hat{e}_2, e_3) \geq 0. \end{aligned}$$

Using Lemma 6, we obtain

$$\begin{aligned} & R^\#(e_1, e_3, e_1, e_3) + \lambda^2 R^\#(e_1, e_4, e_1, e_4) \\ (7) \quad & + \mu^2 R^\#(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R^\#(e_2, e_4, e_2, e_4) \\ & + 2\lambda\mu R^\#(e_1, e_3, e_4, e_2) + 2\lambda\mu R^\#(e_1, e_4, e_2, e_3) \geq 0. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & R^2(e_1, e_3, e_1, e_3) + \lambda^2 R^2(e_1, e_4, e_1, e_4) \\ & + \mu^2 R^2(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R^2(e_2, e_4, e_2, e_4) \\ (8) \quad & + 2\lambda\mu R^2(e_1, e_3, e_4, e_2) + 2\lambda\mu R^2(e_1, e_4, e_2, e_3) \\ & = \sum_{p,q=1}^n [R(e_1, e_3, e_p, e_q) - \lambda\mu R(e_2, e_4, e_p, e_q)]^2 \\ & + \sum_{p,q=1}^n [\lambda R(e_1, e_4, e_p, e_q) + \mu R(e_2, e_3, e_p, e_q)]^2 \geq 0. \end{aligned}$$

Adding (7) and (8), we conclude that

$$\begin{aligned} & Q(R)(e_1, e_3, e_1, e_3) + \lambda^2 Q(R)(e_1, e_4, e_1, e_4) \\ (9) \quad & + \mu^2 Q(R)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 Q(R)(e_2, e_4, e_2, e_4) \\ & + 2\lambda\mu Q(R)(e_1, e_3, e_4, e_2) + 2\lambda\mu Q(R)(e_1, e_4, e_2, e_3) \geq 0. \end{aligned}$$

Since

$$Q(R)(e_1, e_2, e_3, e_4) + Q(R)(e_1, e_3, e_4, e_2) + Q(R)(e_1, e_4, e_2, e_3) = 0,$$

the assertion follows.

Proposition 10. *Suppose that $R(t)$, $t \in [0, T)$, is a solution of the ODE $\frac{d}{dt}R(t) = Q(R(t))$ with $R(0) \in E$. Then $R(t) \in E$ for all $t \in [0, T)$.*

Proof. Fix $\varepsilon > 0$, and denote by $R_\varepsilon(t)$ the solution of the ODE $\frac{d}{dt}R_\varepsilon(t) = Q(R_\varepsilon(t)) + \varepsilon I$ with initial condition $R_\varepsilon(0) = R(0) + \varepsilon I$. The function $R_\varepsilon(t)$ is defined on some time interval $[0, T_\varepsilon)$. We claim that $R_\varepsilon(t) \in E$ for all

$t \in [0, T_\varepsilon)$. To prove this, we argue by contradiction. Suppose that there exists a time $t \in [0, T_\varepsilon)$ such that $R_\varepsilon(t) \notin E$. Let

$$\tau = \inf\{t \in [0, T_\varepsilon) : R_\varepsilon(t) \notin E\}.$$

Clearly, $\tau > 0$ and $R_\varepsilon(\tau) \in \partial E$. By Proposition 7, we can find an orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and real numbers $\lambda, \mu \in [-1, 1]$ such that

$$\begin{aligned} & R_\varepsilon(\tau)(e_1, e_3, e_1, e_3) + \lambda^2 R_\varepsilon(\tau)(e_1, e_4, e_1, e_4) \\ & + \mu^2 R_\varepsilon(\tau)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R_\varepsilon(\tau)(e_2, e_4, e_2, e_4) \\ & - 2\lambda\mu R_\varepsilon(\tau)(e_1, e_2, e_3, e_4) + (1 - \lambda^2)(1 - \mu^2) = 0. \end{aligned}$$

By definition of τ , we have $R_\varepsilon(t) \in E$ for all $t \in [0, \tau)$. This implies

$$\begin{aligned} & R_\varepsilon(t)(e_1, e_3, e_1, e_3) + \lambda^2 R_\varepsilon(t)(e_1, e_4, e_1, e_4) \\ & + \mu^2 R_\varepsilon(t)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R_\varepsilon(t)(e_2, e_4, e_2, e_4) \\ & - 2\lambda\mu R_\varepsilon(t)(e_1, e_2, e_3, e_4) + (1 - \lambda^2)(1 - \mu^2) \geq 0 \end{aligned}$$

for all $t \in [0, \tau)$. Hence, we obtain

$$\begin{aligned} & Q(R_\varepsilon(\tau))(e_1, e_3, e_1, e_3) + \lambda^2 Q(R_\varepsilon(\tau))(e_1, e_4, e_1, e_4) \\ & + \mu^2 Q(R_\varepsilon(\tau))(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 Q(R_\varepsilon(\tau))(e_2, e_4, e_2, e_4) \\ & - 2\lambda\mu Q(R_\varepsilon(\tau))(e_1, e_2, e_3, e_4) + \varepsilon(1 + \lambda^2)(1 + \mu^2) \leq 0. \end{aligned}$$

On the other hand, since $R_\varepsilon(\tau) \in E$, we have

$$\begin{aligned} & Q(R_\varepsilon(\tau))(e_1, e_3, e_1, e_3) + \lambda^2 Q(R_\varepsilon(\tau))(e_1, e_4, e_1, e_4) \\ & + \mu^2 Q(R_\varepsilon(\tau))(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 Q(R_\varepsilon(\tau))(e_2, e_4, e_2, e_4) \\ & - 2\lambda\mu Q(R_\varepsilon(\tau))(e_1, e_2, e_3, e_4) \geq 0 \end{aligned}$$

by Proposition 9. This is a contradiction.

Thus, we conclude that $R_\varepsilon(t) \in E$ for all $t \in [0, T_\varepsilon)$. It follows from standard ODE theory that $T \leq \liminf_{\varepsilon \rightarrow 0} T_\varepsilon$ and $R(t) = \lim_{\varepsilon \rightarrow 0} R_\varepsilon(t)$ for all $t \in [0, T)$. Therefore, we have $R(t) \in E$ for all $t \in [0, T)$. This completes the proof.

As in [1], we define a family of linear transformations $\ell_{a,b}$ on the space of algebraic curvature operators by

$$\ell_{a,b}(R) = R + b \operatorname{Ric}_0 \otimes \operatorname{id} + \frac{a}{n} \operatorname{scal} \operatorname{id} \otimes \operatorname{id}.$$

Here, scal and Ric_0 denote the scalar curvature and trace-free Ricci tensor of R , respectively. Moreover, \otimes denotes the Kulkarni-Nomizu product, i.e.

$$(A \otimes B)_{ijkl} = A_{ik} B_{jl} - A_{il} B_{jk} - A_{jk} B_{il} + A_{jl} B_{ik}.$$

Using a result of C. Böhm and B. Wilking [1], we obtain:

Proposition 11. *Assume that $b \in (0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}]$ and $2a = 2b + (n-2)b^2$. Then the set $\ell_{a,b}(E)$ is invariant under the ODE $\frac{d}{dt}R = Q(R)$.*

Proof. By work of Böhm and Wilking (cf. [1], Theorem 2), it suffices to show that the set E is invariant under the ODE $\frac{d}{dt}R = Q(R) + D_{a,b}(R)$, where $D_{a,b}(R)$ is defined by

$$\begin{aligned} D_{a,b}(R) = & ((n-2)b^2 - 2(a-b)) \operatorname{Ric}_0 \oslash \operatorname{Ric}_0 \\ & + 2a \operatorname{Ric} \oslash \operatorname{Ric} + 2b^2 \operatorname{Ric}_0^2 \oslash \operatorname{id} \\ & + \frac{nb^2(1-2b) - 2(a-b)(1-2b+nb^2)}{n+2n(n-1)a} |\operatorname{Ric}_0|^2 \operatorname{id} \oslash \operatorname{id}. \end{aligned}$$

The first term on the right vanishes as $2a = 2b + (n-2)b^2$. By Corollary 8, E is a subset of \tilde{C} . Hence, every algebraic curvature operator $R \in E$ has nonnegative Ricci curvature. Consequently, we have $D_{a,b}(R) \geq 0$ for all $R \in E$. Since E is invariant under the ODE $\frac{d}{dt}R = Q(R)$ by Proposition 10, we conclude that E is also invariant under the ODE $\frac{d}{dt}R = Q(R) + D_{a,b}(R)$.

4. PROOF OF THE MAIN THEOREM

The proof of Theorem 2 relies on the construction of a suitable pinching set. The concept of a pinching set was introduced in pioneering work of Hamilton (cf. [4], Definition 5.1). Böhm and Wilking [1] have a slightly more general notion of pinching set, which is more convenient for our purposes.

Proposition 12. *Let K be a compact set which is contained in the interior of \tilde{C} . Then there exists a closed, convex, $O(n)$ -invariant set F with the following properties:*

- (i) F is invariant under the ODE $\frac{d}{dt}R = Q(R)$.
- (ii) For each $\delta \in (0, 1)$, the set $\{R \in F : R \text{ is not } \delta\text{-pinched}\}$ is bounded.
- (iii) K is a subset of F .

Proof. By assumption, the set K is contained in the interior of \tilde{C} . Using Proposition 4, we obtain

$$\begin{aligned} & R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ & + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) - 2\lambda\mu R(e_1, e_2, e_3, e_4) > 0 \end{aligned}$$

for all $R \in K$, all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$, and all pairs $(\lambda, \mu) \in \partial([-1, 1] \times [-1, 1])$. Hence, there exists a positive real number N with the following properties:

1. We have

$$\begin{aligned} & R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ & + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) \\ & - 2\lambda\mu R(e_1, e_2, e_3, e_4) + N(1 - \lambda^2)(1 - \mu^2) > 0 \end{aligned}$$

for all $R \in K$, all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$, and all pairs $(\lambda, \mu) \in [-1, 1] \times [-1, 1]$.

2. We have $\operatorname{tr}(R) \leq 2N$ for all $R \in K$.

Without loss of generality, we may assume that $N = 1$. Thus, K is contained in the interior of the set E . Consequently, we can find real numbers $a \in (0, \frac{1}{2(n-1)}]$ and $b \in (0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}]$ such that $2a = 2b + (n-2)b^2$ and $K \subset \ell_{a,b}(E)$. We now define $F_1 = \ell_{a,b}(E)$. Clearly, F_1 is closed, convex, and $O(n)$ -invariant. Moreover, F_1 is invariant under the ODE $\frac{d}{dt}R = Q(R)$ by Proposition 11.

We next consider the cones $\hat{C}(s)$ defined in [2]. By continuity, we can find a real number $s_1 > 0$ such that $\ell_{a,b}(\hat{C}) \subset \hat{C}(s_1)$. Hence, it follows from Corollary 8 that

$$\ell_{a,b}(R) + (1 + 2(n-1)a)I = \ell_{a,b}(R + I) \in \ell_{a,b}(\hat{C}) \subset \hat{C}(s_1)$$

for all $R \in E$. Since $a \in (0, \frac{1}{2(n-1)}]$, we conclude that

$$F_1 \subset \{R : R + 2I \in \hat{C}(s_1)\}.$$

Using Proposition 16 in [2], we can construct an increasing sequence of positive real numbers s_j , $j \in \mathbb{N}$, and a sequence of closed, convex, $O(n)$ -invariant sets F_j , $j \in \mathbb{N}$, with the following properties:

- (a) For each $j \in \mathbb{N}$, we have $F_{j+1} = F_j \cap \{R : R + 2^{j+1}I \in \hat{C}(s_{j+1})\}$.
- (b) For each $j \in \mathbb{N}$, we have $F_j \cap \{R : \text{tr}(R) \leq 2^j\} \subset F_{j+1}$.
- (c) For each $j \in \mathbb{N}$, the set F_j is invariant under the ODE $\frac{d}{dt}R = Q(R)$.
- (d) $s_j \rightarrow \infty$ as $j \rightarrow \infty$.

We now define $F = \bigcap_{j=1}^{\infty} F_j$. Clearly, F is a closed, convex, $O(n)$ -invariant set, which is invariant under the ODE $\frac{d}{dt}R = Q(R)$. Since $K \subset F_1 \cap \{R : \text{tr}(R) \leq 2\}$, it follows from property (b) that $K \subset F_j$ for all $j \in \mathbb{N}$. Hence, K is a subset of F . Finally, property (a) implies

$$F \subset F_j \subset \{R : R + 2^jI \in \hat{C}(s_j)\}$$

for all $j \in \mathbb{N}$. Since $s_j \rightarrow \infty$ as $j \rightarrow \infty$, the assertion follows from Proposition 15 in [2].

Having established the existence of a pinching set, the convergence of the normalized Ricci flow follows from work of Hamilton [4] (see also [1], Theorem 5.1):

Theorem 13. *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$. Assume that the curvature tensor of (M, g_0) lies in the interior of the cone \tilde{C} for all points in M . Then the normalized Ricci flow with initial metric g_0 exists for all time and converges to a metric of constant sectional curvature as $t \rightarrow \infty$.*

By Proposition 4, every curvature tensor satisfying (2) lies in the interior of the cone \tilde{C} . Thus, Theorem 2 is an immediate consequence of Theorem 13.

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